

Bag Models with Fuzzy Boundaries¹

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We discuss how hadronic bag models can be generalized in the framework of fuzzy set theory to implement effects of a smooth and extended phase boundary.

Idealizations in physical models typically arise either from insufficient knowledge of the underlying physics, or from the desire to make them more transparent and amenable to quantitative analysis. Hadronic bag models [2,3] furnish a typical example for both cases. They impose quark confinement inside hadrons, in a region of modified vacuum, by static boundary conditions at a bag radius R . Whereas the real vacuum is expected to return to its normal phase outside of the hadron gradually, this simple prescription leads to an infinitely thin and energetically unfavorable bag boundary, and to an abrupt transition between the two phases.

In the following we will discuss a new implementation of extended boundaries [4] which is easy to apply to even the most complex bag models. This approach is formulated in terms of fuzzy set theory [5,6], in which ordinary sets are generalized by assigning partial memberships to their elements. The application to the transition between the inside and outside regions of bag models offers itself naturally since fuzzy sets were specifically designed to implement smooth transitions between unrealistically distinct domains in mathematical models.

The fuzzy boundary can be most easily envisioned by considering the sharp surface of the standard bag model at a given radius as the sole element of an ordinary set. By letting this set become fuzzy, an extended boundary – containing conventional bag surfaces of varying radii and weights as elements – emerges. In analogy with the boundary conditions of standard bag models, the underlying fuzzy set ρ , the “fuzzy bag radius”, is prescribed according to general physical requirements.

Fuzzy sets [5] consist of an ordinary reference set \mathcal{X} and a real-valued membership function

$$\mu : \mathcal{X} \rightarrow [0, 1] \quad x \mapsto \mu(x), \quad (1)$$

which specifies the degree to which an element $x \in \mathcal{X}$ belongs to μ . (Following common practice, we use the same symbol for both the fuzzy set and its membership function.)

Accordingly, the fuzzy bag radius is represented by a membership function $\rho(R)$, which specifies the degree to which a sphere of radius R belongs to the extended bag boundary. Its reference set $\mathcal{R} \subseteq [0, \infty]$ minimally con-

tains the radii in the surface region. We denote the center (in radial direction) of the boundary by R_0 and its width by Δ . Some of the potential of this description of the boundary derives from the fact [6] that membership degrees in fuzzy sets are generally not additive (in contrast to, e.g., probability measures). This implies that bag surfaces at different R can be correlated and coexisting in a common, extended boundary.

Since bag models do not provide any dynamics for the boundary, we have to rely on more general physical requirements to find the appropriate shape of ρ [4]. The typical surface shapes found in nontopological soliton models [7], in particular, do not show significant asymmetries between the inner and outer parts of the surface. This suggests the use of a Gaussian membership function

$$\rho^{(g)}(R) = \exp \left[\frac{-(R - R_0)^2}{2\Delta^2} \right] \quad (2)$$

for the fuzzy bag radius, to which we will adhere below. In order to check the dependence of the results on the detailed shape of the membership function, we have also tested alternative choices such as the triangular form $\rho^{(t)}(R) = 1 - \left| \frac{R - R_0}{2\Delta} \right|$ for $|R - R_0| \leq 2\Delta$, and $\rho^{(t)}(R) = 0$ otherwise. (Note that $\rho^{(t)} \subseteq \rho^{(g)}$.) In all cases, the standard bag model is recovered for $\Delta \rightarrow 0$.

The next step in the setup of the fuzzy bag model deals with the definition and calculation of observables. Starting from a conventional bag model with crisp bag radius, this is accomplished by employing the extension principle [8] of fuzzy set theory. Adapted to the present context, it states that any map $A(R)$ from a (crisp) bag radius R to an observable $A \in \mathcal{A}$ (as calculated in conventional bag models) can be uniquely extended to a map from the fuzzy bag radius $\rho(R)$ to a fuzzy set

$$\begin{aligned} \nu : \mathcal{F}_I(\mathcal{R}) &\rightarrow \mathcal{F}(\mathcal{A}), & \rho(\mathcal{R}) &\mapsto \nu_\rho(\mathcal{A}) \\ \nu_\rho(x) &:= \sup \{ \rho(R) \mid R \in \mathcal{R} \wedge x = A(R) \}. \end{aligned} \quad (3)$$

Equation (3) quantifies how the fuzziness of the basic variable R propagates into the observables. It follows directly from the rules which govern fuzzy sets [8].

In order to convert fuzzy-bag results, i.e. the fuzzy sets $\nu(A)$, into numerical predictions, they have to be mapped onto those real numbers \tilde{A} which best represent their physical information content. For this purpose we employ the standard centroid map [9]

$$\tilde{A} = \frac{\int A \nu(A) dA}{\int \nu(A) dA}. \quad (4)$$

(The integrals extend over \mathcal{A} .) In subsequent calculations, the fuzzy results $\nu(A)$ should be manipulated directly whenever the involved mathematical operations can be extended to fuzzy intervals.

The above steps complete the definition of the fuzzy bag model as the most direct and transparent fuzzy-set extension of the standard bag model. We now apply these concepts to the nonlinear chiral bag model of Ref. [3] and select two results which illustrate characteristic properties of the fuzzy extension. The bag-radius dependence of the total bag energy E in the hedgehog state is indicated in Fig. 2 (dotted line). In the corresponding fuzzy bag model, $E(R)$ extends to the fuzzy set

$$\epsilon_\rho(x) = \sup \{ \rho(R) \mid R \in \mathcal{R}_\epsilon \wedge x = E(R) \}, \quad (5)$$

which is plotted in Fig. 1a for $R_0 = 0.7$ fm, $\Delta = 0.3$ fm, and $\mathcal{R}_\epsilon = [0, 1.5]$ fm.

As a second example, we consider the nucleon's axial coupling g_A to first order in the angular velocity Ω [3]. It is plotted in Fig. 3 (dotted line) and shows a significantly stronger bag-radius dependence than the hedgehog energy, which implies a strong deviation from “Cheshire-Cat” behavior (see below). The corresponding fuzzy set

$$\gamma_\rho(x) = \sup \{ \rho(R) \mid R \in \mathcal{R}_\gamma \wedge x = g_A(R) \} \quad (6)$$

is shown in Fig. 1b for $\mathcal{R}_\gamma = [0, 1]$ fm with R_0 and Δ as above. The shapes of ϵ and γ closely reflect the behavior of $E(R)$ and $g_A(R)$ and share some general properties of fuzzy bag-model observables, as discussed in Ref. [4].

Next, we calculate the centroids of $\epsilon(E)$ and $\gamma(g_A)$ according to Eq. (4) and examine the dependence of the resulting fuzzy-bag observables \tilde{E} and \tilde{g}_A on location and extension of the boundary region. Figure 2 shows the hedgehog energy \tilde{E} as a function of R_0 for different values of the “fuzziness” parameter Δ . (In the following, we drop the tilde on fuzzy-bag results and identify them by their R_0 -dependence.) The dotted line corresponds to $\Delta \rightarrow 0$.

For increasing boundary diffuseness, the bag energy becomes less sensitive to R_0 until, beyond $\Delta \sim 0.4$ fm, it remains almost R_0 -independent. With $\Delta = 0.4$ fm and for R_0 in the range $0 \leq R_0 \leq 1$ fm, e.g., g_A deviates less than 10 % from its experimental value 1.26 [10]. Furthermore, the extended boundary shifts the minimum of the fuzzy-bag energy towards smaller radii, from 0.85 to 0.5 fm. The reduced sensitivity of fuzzy bag model results to the boundary position reduces the parameter dependence of the model and complies with the Cheshire-Cat principle [11]. Indeed, exact Cheshire-Cat models are fixed points under fuzzification [4].

In order to get an idea of the model dependence associated with different (e.g., triangular and Gaussian) boundary shapes, one may use the fuzzy measure [9]

$$\| \mu_1 = \mu_2 \| = \inf \{ 1 - |\mu_1(x) - \mu_2(x)| \mid x \in \mathcal{X} \} \quad (7)$$

for the equality of two fuzzy sets μ_1, μ_2 , which yields $\| \rho^{(g)} = \rho^{(t)} \| \simeq 0.9$ and $\| \epsilon^{(g)} = \epsilon^{(t)} \| \simeq 0.85$, $\| \gamma^{(g)} = \gamma^{(t)} \| \simeq 0.9$ (almost independently of Δ), and indicates the predictions to be rather robust.

In summary, fuzzy bag models incorporate effects of a smooth phase boundary in a simple and rather unbiased way, while maintaining the appealing simplicity and the absolute confinement of conventional bag models. The first results are encouraging and provide motivation for further conceptual developments and the exploration of potential links, e.g., to fuzzy spaces and noncommutative geometry. Moreover, it seems likely that interesting physical applications for fuzzy sets beyond the realms of the bag model and hadronic physics can be found.

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FIG. 1. The membership function of a) the bag energy and b) the nucleon's axial coupling ($R_0 = 0.7$ fm, $\Delta = 0.3$ fm).

FIG. 2. The bag energy for $\Delta = 0$ fm (dotted line), 0.1 fm (dashed), 0.2 fm (dot-dashed), 0.3 fm (dot-dot-dashed), 0.4 fm (solid).

FIG. 3. The nucleon's axial coupling for the values of Δ as above.

Fig. 1a

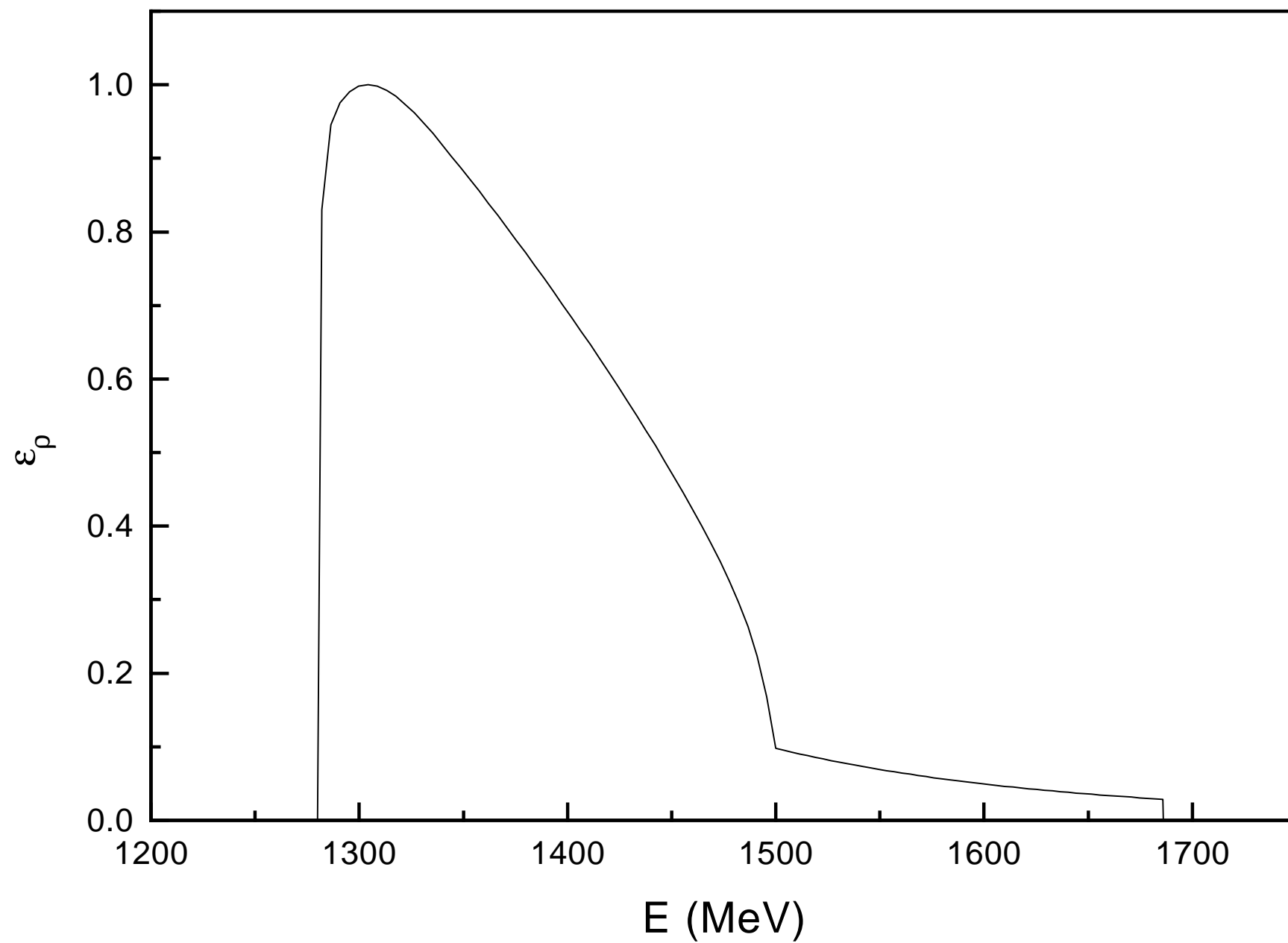


Fig. 2

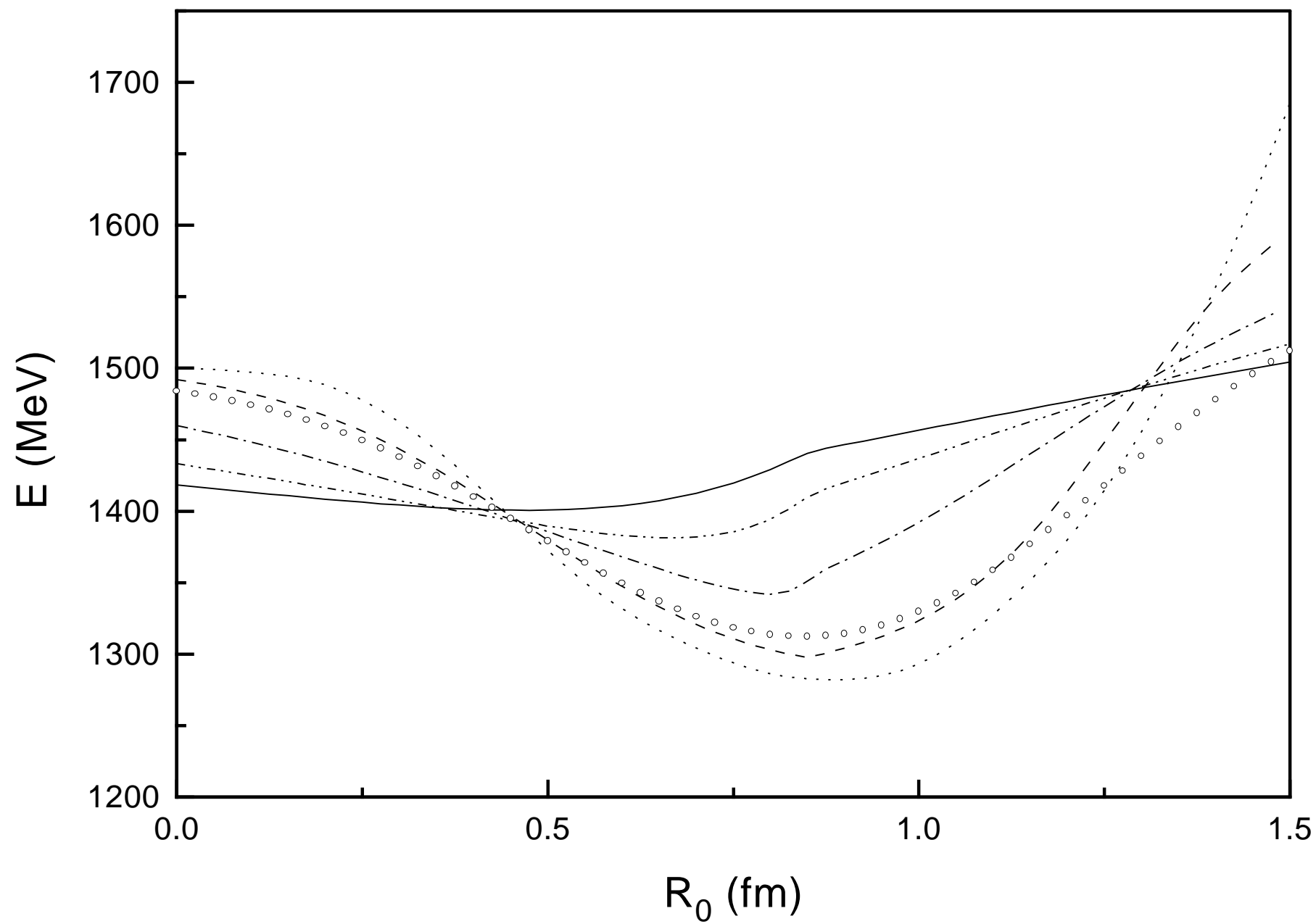


Fig. 3

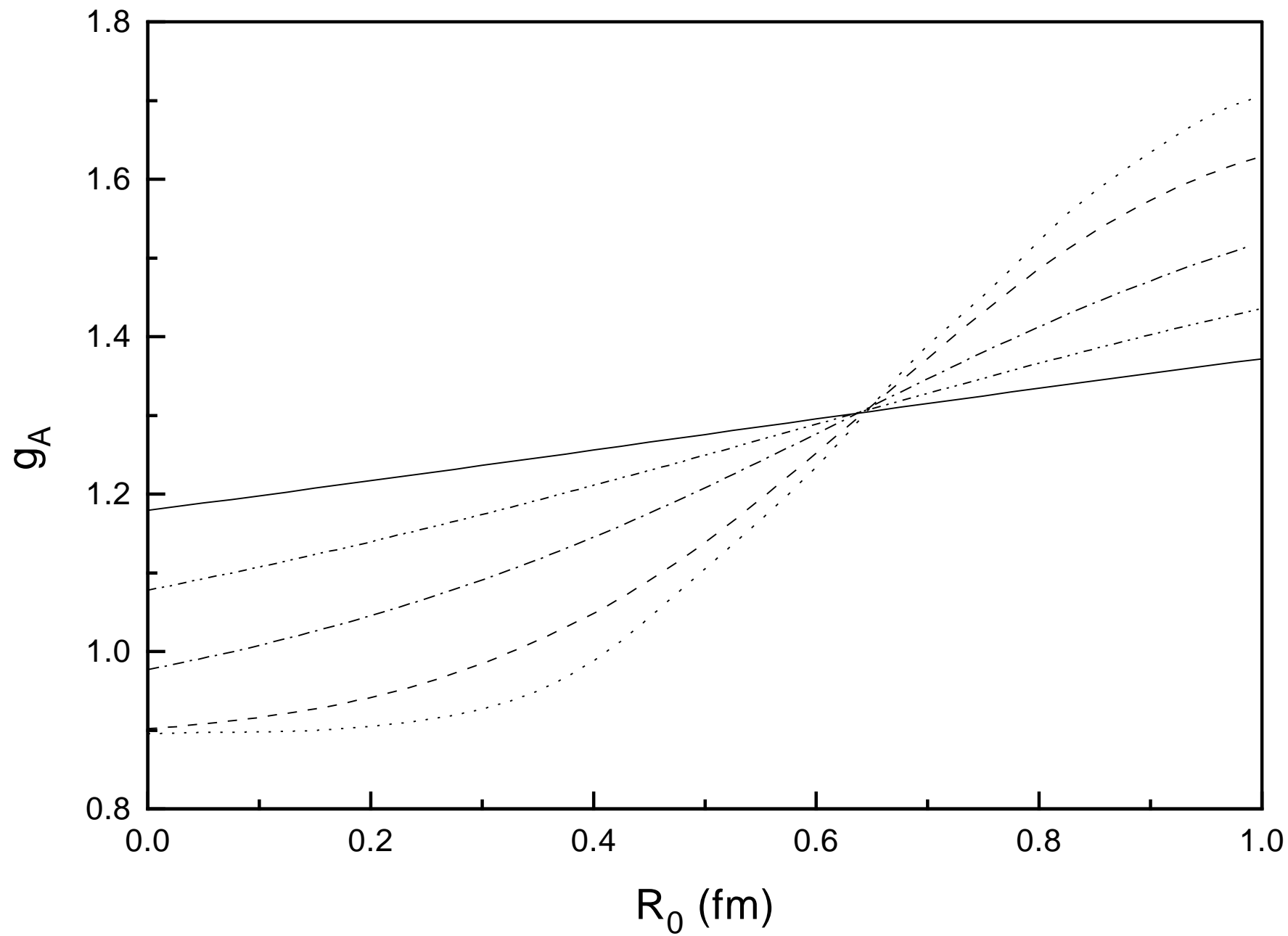


Fig. 1b

